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FINITELY-PRESENTED GROUPS WITH LONG LOWER CENTRAL SERIES

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ABSTRACT

We construct examples of finitely presented groups G with the property that, if G_ω is the intersection of the lower central series of G, then $[G,G_\omega] \neq G_\omega$.

The transfinite lower central series $\{G_\alpha\}$ of a group G is defined, by transfinite induction, for every ordinal α (less than some fixed ordinal λ) as follows:

(1) $G_1 = G$,

(2) $G_{\alpha+1} = [G_\alpha, G]$ for any ordinal α,

(3) $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$, if α is a limit ordinal.

(If H,H' are subgroups of G, then $[H,H']$ is the subgroup generated by all commutators $[h,h']$, $h \in H$, $h' \in H'$.) It is clear that this series eventually stabilizes, i.e., $G_\alpha = G_{\alpha+1}$ if α is large enough (also λ). The smallest such α will be referred to as the *length* of G. If $G_\alpha = \{1\}$, G is *transfinitely nilpotent.*

It is a classical theorem of Malcev [M] that, for any ordinal α, there exists a group of length α. Furthermore, it is shown by Hall and Hartley [HH] (see also [H] and [H1]) that, if α is countable, there is a finitely-generated group of length α. It is suggestive that all the examples constructed in these papers, when $\alpha > \omega$ (ω is the first infinite ordinal), are non-finitely-presented. Thus it is natural to ask what lengths are possible for finitely-presented groups. In particular, is it possible for a finitely-presented group to have length greater than ω? Note that any non-abelian free group has length ω [Ma] and it is known [H] that any extension of a finitely-generated abelian group by a cyclic group has length $\leq\omega$.

It is the main purpose of this note to prove:

THEOREM 1. *There exists a finitely-presented group G of length $>\omega$, i.e. $G_\omega \neq G_{\omega+1}$.*

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The proof will make use of a new localization process for groups ([L1], [V]), used in [L], [Le] to study concordance of links. This is a substitute for the *HZ-localization* of Bousfield ([B]) and is better suited to the study of objects of finite type (also see [FOS]).

Theorem 1 will follow from the existence of a group whose localization is not "achieved" in the nilpotent completion (see Theorem 2).

In the final section we will produce an explicit group G satisfying Theorem 1, whose construction is suggested by Theorem 2, and give a direct proof (without using localization) that $G_\omega \neq G_{\omega+1}$.

1. We recall some definitions from [L], [L1]. Let G be a group and $\mathcal{S}$ a *system of equations* over G:

$$(\mathcal{S}) \qquad x_i = w_i(x_1, \ldots, x_n), \qquad 1 \le i \le n.$$

$w_i = w_i(x_1, \ldots, x_n)$ is an element of $G * F$ (free product), where F is the free group on the indeterminants $x_1, \ldots, x_n$. A *solution* of $\mathcal{S}$ is a retraction $\rho : G * F \to G$ such that $\rho(w_i) = \rho(x_i)$, $1 \le i \le n$. We usually write the solution, more suggestively, as "$x_i = \rho(w_i)$." We say $\mathcal{S}$ is *contractible* if $\rho'(w_i) = 1$ for all i, where $\rho' : G * F \to F$ is the retraction defined by $\rho'(G) = 1$. G is *E-closed* (called algebraically closed in [L]) if every contractible system of equations over G has a *unique* solution in G.

It is proved in [L] that, for every group G, there is an essentially unique $\phi : G \to \hat{G}$ satisfying:

(i) $\hat{G}$ is E-closed.

(ii) If $\psi : G \to A$, where A is E-closed, then there is a unique $\theta : \hat{G} \to A$ such that $\theta \circ \varphi = \psi$. $\hat{G}$ is called the *algebraic closure* of G. We refer the reader to [L], [L1] for properties and examples.

Suppose $\psi : G \to A$ is as in (ii). Then we can define the *algebraic closure of* G (or ψ) *in* A to be $\theta(\hat{G})$. As our main example of this let $A = \tilde{G}$, the *nilpotent completion* of G ($\tilde{G} = \lim_{q \to \infty} G/G_q$) and ψ the canonical homomorphism. It is proved in [L2] that $\tilde{G}$ is E-closed. We denote by $\bar{G}$ the algebraic closure of G in $\tilde{G}$. We will see below that $\operatorname{Ker}\{\hat{G} \to \bar{G}\} = (\hat{G})_\omega$. It is an important question in link theory whether $\hat{F} = \bar{F}$, or, at least, length $\hat{F} = \omega$, where F is any free group (see question 4 in section 5 of this paper). We shall prove:

THEOREM 2. *There exist finitely presented groups G such that* length $\hat{G} > \omega$.

The examples we exhibit are the following family for $n \geq 2$:

$$\Gamma(n) = \{t, x_1, \ldots, x_n : x_i x_j = x_j x_i \ (1 \leq i < j \leq n),\ t x_i t^{-1} = x_i^{-1} \ (1 \leq i \leq n)\}.$$

We shall also show that Theorem 2 implies Theorem 1.

2. We will have use for a tower construction, modeled after a similar construction in [B], first suggested by Kent Orr, which produces the transfinite lower central series of $\hat{G}$. We construct, for any group G, an inverse system $\{G(\alpha);$ $\varphi_{\alpha\beta}: G(\alpha) \to G(\beta)$, if $\alpha > \beta$; $\varphi_\alpha : G \to G(\alpha)\}$ where the indexing set is all ordinals (less than a given ordinal). Set $G(1) = 1$ and $G(2) = H_1(G)$. For $\alpha > 2$ proceed by induction and assume $\{G(\beta), \phi_\beta, \phi_{\beta\gamma}\}$ is defined for all $\gamma < \beta < \alpha$. If $\alpha = \beta + 1$, then $\phi_{\alpha\beta}: G(\alpha) \to G(\beta)$ is defined to be the central extension defined by

$$H_2 G(\beta) \to \mathrm{Cok}\{\varphi_{\beta *} : H_2 G \to H_2 G(\beta)\}$$

and $\varphi_\alpha : G \to G(\alpha)$ is the canonical lift of φ_β (see [B]). If α is a limit ordinal, then $G(\alpha)$ is defined to be the algebraic closure of G in $\lim_{\beta<\alpha} G(\beta)$. We rely on the facts that a central extension of an E-closed group is E-closed, and the inverse limit of a system of E-closed groups is E-closed (see [L]) to insure that $G(\alpha)$ is well-defined and E-closed. Thus, by the definition of $\hat{G}$, φ_α lifts uniquely to $\hat{\varphi}_\alpha : \hat{G} \to G(\alpha)$.

PROPOSITION 1. *$\hat{\varphi}_\alpha$ is onto and* Kernel $\hat{\varphi}_\alpha = (\hat{G})_\alpha$.

REMARK. The tower construction in [B] is identical to ours, except that $G(\alpha)$ is the HZ-closure of $\lim_{\beta<\alpha} G(\beta)$, if α is a limit ordinal. According to [B], $G(\alpha)$ is the α-th lower central series quotient of the HZ-localization of G (which is always transfinitely nilpotent, unlike the algebraic closure).

PROOF. Recall from [B] the tower construction of the lower central series quotients of G. $G[1] = G$, $G[\alpha + 1] \to G[\alpha]$ is the central extension defined by $H_2 G[\alpha] \to \mathrm{Cok}\{H_2 G \to H_2 G[\alpha]\}$, and $G[\alpha] =$ image of G in $\lim_{\beta<\alpha} G[\beta]$ if α is a limit ordinal. Then $G[\alpha] \approx G/G_\alpha$ in a natural way. We apply this tower construction to $\hat{G}$ and show that there is an isomorphism of inverse systems $\hat{G}[\alpha] \approx G(\alpha)$, by transfinite induction. If $\alpha = \beta + 1$ and $\hat{G}[\beta] \approx G(\beta)$, then $\hat{G}[\alpha]$ and $G(\alpha)$ are the central extension of $G(\beta)$ defined by

$$H_2 \hat{G}[\beta] \to \mathrm{cok}\{H_2 \hat{G} \to H_2 \hat{G}[\beta]\} \quad \text{and} \quad H_2 G(\beta) \to \mathrm{cok}\{H_2 G \to H_2 G(\beta)\},$$

respectively. Since $H_2 G \to H_2 G$ is onto, these are the same. If α is a limit ordinal, the inductive hypothesis implies $\lim_{\beta<\alpha} \hat{G}[\beta] \approx \lim_{\beta<\alpha} G(\beta)$. Then $\hat{G}[\alpha]$ is the image of $\hat{G}$, while $G(\alpha)$ is the algebraic closure of the image of G in $\lim_{\beta<\alpha} G(\beta)$. But these are identical by definition.

This completes the proof.

COROLLARY. *For any group G,* Kernel$\{\hat{G} \to \bar{G}\} = (\hat{G})_\omega$. *(Recall $\bar{G}$ is the algebraic closure of G in its nilpotent completion $\tilde{G}$.)*

3. We show now that Theorem 2 implies Theorem 1. Suppose G is a finitely-presented group with length $\hat{G} > \omega$. It follows from the Stallings 5-term exact sequence [S] that this is equivalent to $H_2(\hat{G}) - H_2(\hat{G}/(\hat{G})_\omega)$, not onto. Now, according to [L], $\hat{G}$ is a direct limit of finitely-presented groups:

$$G = G^0 \to G^1 \to \cdots \to G^r \to G^{r+1} \to \cdots,$$

where $G \to G^r$ is 2-connected, for all r, i.e., $H_q(G) \to H_q(G^r)$ is bijective for $q = 1$ and surjective for $q = 2$. By [S], $G \to (\tilde{G}^r)$ is bijective and so we obtain a sequence of inclusions:

$$G/G_\omega \subseteq \cdots \subseteq G^r/(G^r)_\omega \subseteq / G^{r+1}/(G^{r+1})_\omega \subseteq \cdots \subseteq \hat{G}/(\hat{G})_\omega .$$

Clearly $\hat{G}/(\hat{G})_\omega = \bigcup_r G^r/(G^r)_\omega$ and so $H_2(\hat{G}/(\hat{G})_\omega) = \lim_{r\to\infty} H_2(G^r/(G^r)_\omega)$. We can now conclude that $H_2(G^r) \to H_2(G^r/(G^r)_\omega)$ is not onto for some r, for otherwise

$$H_2(\hat{G}) \approx \lim_r H_2(G^r) \to \lim_r H_2(G^r/(G^r)_\omega) \approx H_2(\hat{G}/\hat{G})_\omega)$$

would be onto. For such G^r we then conclude by [S] that length $G^r > \omega$.

4. We now prove Theorem 2 by exhibiting a finitely-presented group G such that $H_2(G) \to H_2(\bar{G})$ is not onto. Since $H_2(G) \to H_2(\hat{G})$ is onto and $\bar{G} = \hat{G}/(\hat{G})_\omega$, this will show that $H_2(\hat{G}) \to H_2(\hat{G}/(\hat{G})_\omega)$ is not onto and so length $\hat{G} > \omega$.

Let G be the semi-direct product of a finitely-generated abelian group A by an infinite cyclic group C, i.e. A is a normal subgroup of G and $G/A \approx C$. G is then completely specified by the conjugating action of C on A. If $t \in G$ is a generator of C, we define $t\alpha t^{-1} = \alpha^{-1}$ for all $\alpha \in A$.

Let Z_2 denote the additive group of 2-adic integers and $Z_{(2)}$ ($\subseteq Z_2$) the rational 2-adic integers. We now define $A_2 = A \otimes Z_2$ and $A_{(2)} = A \otimes Z_{(2)}$, interpreting A additively.

PROPOSITION 2. *$\tilde{G}$ and $\bar{G}$ are the semi-direct products of A_2 and $A_{(2)}$, respectively, by C, with $t\alpha t^{-1} = \alpha^{-1}$ for all $\alpha \in A_2$ or $A_{(2)}$. The inclusion $\bar{G} \subseteq \tilde{G}$ is induced by the natural inclusion $A_{(2)} \subseteq A_2$.*

PROOF. It is straightforward to calculate that the lower central series of G is given by:

$$G_q = 2^{q-1}A \qquad \text{for } q \geq 2$$

using additive notation in A. Thus the result for $\tilde{G}$ follows immediately. We now have to identify $\bar{G}$ in $\tilde{G}$.

A typical element of $\bar{G}$ occurs as a solution of a contractible system of equations $\mathcal{S}$:

$$(\mathcal{S}) \qquad x_i = w_i(x_1, \ldots, x_n); \qquad 1 \leq i \leq n.$$

Let us formally set $x_i = y_i t^{e_i}$, where $\{y_i\}$ are new indeterminants and e_i are integer variables. We identify t with the generator of C and regard $y_i \in A_2$; so $\mathcal{S}$ is an equation in $\tilde{G}$. Making this substitution in $\mathcal{S}$ and using the relations $ty_i t^{-1} = y_i^{-1}$ we rewrite

$$w_i(x_1, \ldots, x_n) = v_i(y_1, \ldots, y_n)t^{a_i}$$

where $v_i \in A * F(y_1, \ldots, y_n)$. Since $\mathcal{S}$ is contractible and conjugating by t does not change the exponent of y_i mod 2, it follows that the image of v_i under the projection $A * F(y_1, \ldots, y_n) \to F(y_1, \ldots, y_n)$ is a monomial in which the exponent sum of each y_j is even.

This substitution has converted $\mathcal{S}$ into a new system of equations over G:

$$(\mathcal{S}') \qquad y_i t^{e_i} = v_i(y_1, \ldots, y_n)t^{a_i}; \qquad 1 \leq i \leq n.$$

Since $\mathcal{S}'$ is solvable in $\tilde{G}$, we conclude that $a_i = e_i$ and we are left with the following system over A:

$$(\mathcal{S}'') \qquad y_i = v_i(y_1, \ldots, y_n)$$

which is solvable in A_2. If we change to additive notation, $\mathcal{S}''$ can be rewritten as:

$$(\mathcal{S}''') \qquad y_i = \sum_{j=1}^{n} a_{ij} y_j + \alpha_i; \qquad 1 \leq i \leq n$$

where $a_{ij} \in Z$, $\alpha_i \in A$. Our observation above about v_i is equivalent to the condition that each a_{ij} is even. Since square integral matrices with odd determinant are invertible over $Z_{(2)}$, we see that the solutions $\{y_i\}$ of $\mathcal{S}'''$ lie in $A_{(2)}$.

Conversely, suppose $\beta \in A_{(2)}$ and r any integer. We may write $m\beta = \alpha \in A$ for some odd integer $m = 2k + 1$. Then the single contractible equation

$$x = \alpha(xt^{-r})^{-k}t(xt^{-r})^k t^{r-1}$$

has the solution $x = \beta t^r$ (multiplicatively).

This completes the proof of Proposition 2.

5. We now compute $H_2G \to H_2\bar{G}$. Since G is an extension of A by an infinite cyclic group we have the usual exact sequence

$$\cdots \to H_q(A) \xrightarrow{t_*-1} H_q(A) \to H_q(G) \to H_{q-1}(A) \xrightarrow{t_*-1} H_{q-1}(A) \to$$

where t_* is induced by conjugation by t. If A is a *free* abelian group, then $H_q(A) \approx \Lambda^q A$, the q-th exterior power. In our case conjugation by t is inversion and so $t_* = (-1)^q$ on $H_q(A)$. In particular we conclude that $H_2(G) \approx \Lambda^2 A$; similarly $H_2(\bar{G}) \approx \Lambda^2 A_{(2)}$. Furthermore, $H_2(G) \to H_2(\bar{G})$ coincides with the natural homomorphism $\Lambda^2 A \to \Lambda^2 A_{(2)}$. If $\operatorname{rank} A = n$, then $\Lambda^2 A$ (resp. $\Lambda^2 A_{(2)}$) is a direct sum of $\binom{n}{2}$ copies of Z (resp. $Z_{(2)}$). Thus if $n = 1$, $H_2(G) \to H_2(\bar{G})$ is onto, but if $n > 1$ it is not onto and we obtain the desired examples.

This completes the proof of Theorem 2.

6. We now produce an explicit group G satisfying Theorem 1. Let

$$H = \{t, x, y : xy = yx,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1}\}$$

be one of the groups satisfying Theorem 2. Define G from H by adding a generator z and relation $z = x[t,z]$. Let $\phi : H \to G$ be the inclusion and $\psi : G \to H$ the epimorphism defined by:

$$\psi(t) = t, \quad \psi(y) = y, \quad \psi(z) = x, \quad \psi(x) = x^3.$$

Note that ϕ is 2-connected since G is a non-singular extension in the sense of Howie [Ho]. Thus by [S] ϕ induces an isomorphism of nilpotent completions $\tilde{H} \to \tilde{G}$. It is also true that $\psi \circ \phi$ induces an isomorphism of nilpotent completions $\tilde{H} \to \tilde{H}$ (even though $\psi \circ \phi$ is not 2-connected). This can be checked directly from Proposition 2. Thus we conclude that ψ induces an isomorphism $\tilde{G} \to \tilde{H}$. Since ψ is onto, it follows that $G/G_\omega \to H/H_\omega$ (induced by ψ) is an isomorphism.

Now consider the diagram:

$$\begin{array}{ccccccc}
H_2(G) & \to & H_2(G/G_\omega) & \to & G_\omega/G_{\omega+1} & \to & 0 \\
\downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi''} & & \downarrow & & \\
H_2(H) & \to & H_2(H/H_\omega) & \to & H_\omega/H_{\omega+1} & \to & 0
\end{array}$$

where the rows are part of the 5-term exact sequence of [S] arising from the short exact sequence $1 \to G_\omega \to G \to G/G_\omega \to 1$, $1 \to H_\omega \to H \to H/H_\omega \to 1$, and the vertical maps are induced by ψ. Now $H_\omega = 1$ (see Proposition 2) and we see easily that ψ' has cokernel of order 3. In fact, $H_2(H)$ is infinite cyclic generated by $x \wedge y$

and, since ϕ is 2-connected, $H_2(G)$ is generated by $x \wedge y$. Therefore the image of ψ' is generated by $x^3 \wedge y = (x \wedge y)^3$.

Now a simple diagram chase shows that $G_\omega/G_{\omega+1}$ is the group of order 3 — generated by $[z,y]$ — since ψ'' is an isomorphism, $[z,y] = 1$ in G/G_ω, and $\psi''[z,y] = [x,y]$.

7. We conclude with some questions/problems:

(1) Find a finitely-presented group G with length greater than ω, which is transfinitely nilpotent.

(2) Is there a finitely-presented group of any prescribed countable length (and transfinitely nilpotent)?

(3) If G is finitely-generated of length α, is G_α the union of invisible subgroups? Note that G_α is the maximal subgroup K of C satisfying $[K,G] = K$. For the infinitely generated perfect group

$$P = \{x_1, x_2, \ldots, : x_i = [x_{i+1}, x_{i+2}],\ i \geq 1\}$$

we have $P = P_\alpha$ but P contains no invisible subgroups. The finitely-presented group:

$$G = \{x, y, s, t : txt^{-1} = x[x,y] = sxs^{-1},\ tyt^{-1} = y[y, txt^{-1}] = sys^{-1}\}$$

has length ω, but G_ω is normally generated by $\{[s^i t^{-i}, x],\ i > 0\}$ and is not invisible. However, G_ω is the union of the invisible subgroups

$$K(n) = \text{normal closure of } \{[s^i t^{-i}, x],\ i \leq n\}.$$

(4) Is there a finitely-presented group G with length $> \omega$, $H_2(G; Z/p) = 0$ for all p? If not, then $\hat{F}$ has length $= \omega$, where F is the free group of the same rank m as $H_1(G)$, and this implies $H_2(\bar{F}) = 0$. This is important in the study of classical links. If $H_2(\bar{F}) = 0$, then a theorem of X. S. Lin applies to give a complete realizability criterion for Milnor's $\bar{\mu}$-invariants (see [L3]) for links of multiplicity m.

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